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Limit periodic linear difference systems with coefficient matrices from commutative groups

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Abstract. In this paper, limit periodic and almost periodic homogeneous linear difference systems are studied. The coefficient matrices of the considered systems belong to a given commutative group. We find a condition on the group under which the systems, whose fundamental matrices are not almost periodic, form an everywhere dense subset in the space of all considered systems. The treated problem is discussed for the elements of the coefficient matrices from an arbitrary infinite field with an absolute value. Nevertheless, the presented results are new even for the field of complex numbers.

Keywords: limit periodicity, almost periodicity, almost periodic sequences, almost periodic solutions, linear difference equations.

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1 Introduction

For a given commutative group $\mathcal{X}$, we intend to analyse the homogeneous linear difference systems

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z}, \quad \text{where} \quad \{A_k\} \subseteq \mathcal{X}. \tag{1.1}$$

We will consider limit periodic and almost periodic systems (1.1), which means that the sequence of A_k will be limit periodic or almost periodic. The basic motivation of this paper comes from [29, 35].

In [29] (see also [26]), there are studied systems (1.1) for $\mathcal{X}$ being the unitary group and there is proved that, in any neighbourhood of an almost periodic system (1.1), there exist almost periodic systems (1.1) whose fundamental matrices are not almost periodic. The corresponding result about orthogonal difference, skew-Hermitian and skew-symmetric differential systems can be found in [30], [32], and in [34] (see also [27]), respectively. For results concerning almost periodic solutions, we refer to [16, 17, 28, 30], where unitary, orthogonal, skew-Hermitian, and skew-symmetric systems are analysed. In our previous works [13, 33],

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the above mentioned result of [29] is improved for a general (weakly) transformable group $\mathcal{X}$. We remark that the process from [29] cannot be applied for commutative groups of coefficient matrices which are treated in this paper.

In [35], the study of non-almost periodic solutions of limit periodic systems (1.1) has been initiated and the so-called property P has been introduced. The concept of groups with property P leads to results of the same type as the main results of [13, 33]. It should be noted that only bounded groups of matrices are treated in [35]. The goal of this paper is to prove for other groups of matrices that, in any neighbourhood of a system (1.1), there exist systems (1.1) which have at least one non-almost periodic solution. Moreover, we deal with the corresponding Cauchy problems. For this purpose, we generalize the notion of property P (we introduce property P with respect to a given non-trivial vector) and we use the generalization to obtain the announced results for groups which can be unbounded. Especially, for the used modification of property P, it holds that any group which contains a group with the innovated property has this property as well.

The fundamental properties of limit periodic and almost periodic sequences and functions can be found in a lot of monographs (see, e.g., [4, 10, 18, 24]). Almost periodic solutions of almost periodic linear difference equations are studied in articles [6, 7, 8, 12, 14, 37]. Properties of complex almost periodic systems (1.1) are discussed, e.g., in [3, 15, 23]. In the situation when index k attains only positive values, linear almost periodic equations are treated, e.g., in [1, 25]. To the best of our knowledge, the first result about non-almost periodic solutions of homogeneous linear difference equations was obtained in [11].

We prove the announced results using constructions of limit periodic sequences. This approach is motivated by the continuous case (special constructions of homogeneous linear differential systems with almost periodic coefficients are used, e.g., in [19, 20, 21, 22, 32, 34]). Note that the process applied in this paper is substantially different from the ones in all above mentioned papers. Hence, we obtain new results even for almost periodic systems and bounded groups of coefficient matrices.

This paper is organized as follows. In the next section, we mention the notation which is used throughout the whole paper. Then, in Section 3, we define limit periodic, almost periodic, and asymptotically almost periodic sequences and we state their properties which we will need later. In Section 4, we treat the considered homogeneous linear difference systems, where we recall the definitions and results which motivate our recent research and which give the necessary background of the studied problems. In the final section, we formulate and prove our results which are commented by several remarks.

2 Preliminaries

At first, we mention the used notation which is similar to the one from [35]. For arbitrary $p \in \mathbb{N}$, we put $p\mathbb{N} := \{pj : j \in \mathbb{N}\}$. Let $(F, \oplus, \odot)$ be an infinite field. Let $|\cdot|\colon F \to \mathbb{R}$ be an absolute value on F; i.e., let

(i) $|f| \geq 0$ and $|f| = 0$ if and only if f is the zero element,

(ii) $|f \odot g| = |f| \cdot |g|$,

(iii) $|f \oplus g| \leq |f| + |g|$

for all $f, g \in F$.

Let $m \in \mathbb{N}$ be arbitrarily given (as the dimension of later considered systems). The symbol $Mat(F, m)$ will denote the set of all $m \times m$ matrices with elements from F and F^m the set of all $m \times 1$ vectors with elements from F. As usual, the symbols $\cdot$, $+$ will stand for the multiplication and addition on spaces $Mat(F, m)$ and F^m. In $Mat(F, m)$, the identity matrix will be denoted as I and the zero matrix as O.

The absolute value on F gives the norm $\| \cdot \|$ on F^m and $Mat(F, m)$ as the sum of m and m^2 non-negative numbers which are the absolute values of elements, respectively. Especially (consider (ii), (iii)), we have

(I) $\| M + N \| \leq \| M \| + \| N \|$,

(II) $\| u + v \| \leq \| u \| + \| v \|$,

(III) $\| M \cdot N \| \leq \| M \| \cdot \| N \|$,

(IV) $\| M \cdot u \| \leq \| M \| \cdot \| u \|$

for all $M, N \in Mat(F, m)$ and $u, v \in F^m$.

The absolute value on F and the norms on F^m, $Mat(F, m)$ induce the metrics. For simplicity, we will denote each one of these metrics by ϱ. The ε-neighbourhoods will be denoted by $\mathcal{O}^{\varrho}_{\varepsilon}$ in all above given spaces (with metric ϱ). We remark that the metric space (F, ϱ) does not need to be complete or separable (in contrast to [35]).

3 Generalizations of pure periodicity

In this section, we recall the concept of limit periodicity, almost periodicity, and asymptotic almost periodicity for a general metric space (S, τ).

Definition 3.1. We say that a sequence $\{\varphi_k\}_{k \in \mathbb{Z}} \subseteq S$ is *limit periodic* if there exists a sequence of periodic sequences $\{\varphi^n_k\}_{k \in \mathbb{Z}} \subseteq S$, $n \in \mathbb{N}$, such that $\lim_{n \to \infty} \varphi^n_k = \varphi_k$, where the convergence is uniform with respect to $k \in \mathbb{Z}$.

Remark 3.2. Note that limit periodic sequences can be equivalently introduced in a different way. We refer to [5] (see also [2]).

Definition 3.3. We say that a sequence $\{\varphi_k\}_{k \in \mathbb{Z}} \subseteq S$ is *almost periodic* if, for any $\varepsilon > 0$, there exists $r(\varepsilon) \in \mathbb{N}$ such that any set consisting of $r(\varepsilon)$ consecutive integers contains at least one number l satisfying

$$\tau\left(\varphi_{k+l}, \varphi_k\right) < \varepsilon, \quad k \in \mathbb{Z}.$$

The above number l is called an *ε-translation number* of $\{\varphi_k\}$.

Remark 3.4. It is seen directly from Definition 3.3 that any almost periodic sequence is bounded.

Theorem 3.5. *The uniform limit of almost periodic sequences is almost periodic.*

Proof. The theorem can be proved by a simple modification of the proof of [9, Theorem 6.4]. □

Theorem 3.6. *Let $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq S$ be given. The sequence $\{\varphi_k\}$ is almost periodic if and only if any sequence $\{l_n\}_{n\in\mathbb{N}} \subseteq \mathbb{Z}$ has a subsequence $\{\bar{l}_n\}_{n\in\mathbb{N}} \subseteq \{l_n\}$ such that, for any $\varepsilon > 0$, there exists $K(\varepsilon) \in \mathbb{N}$ satisfying*

$$\tau\left(\varphi_{k+\bar{l}_i}, \varphi_{k+\bar{l}_j}\right) < \varepsilon, \quad i,j > K(\varepsilon),\, k \in \mathbb{Z}.$$

Proof. See, e.g., [31, Theorem 2.3]. □

Corollary 3.7. *Let $p \in \mathbb{N}$ be arbitrarily given and let $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq S$ be almost periodic. For any $\varepsilon > 0$, the set of all ε-translation numbers $l \in p\mathbb{N}$ of $\{\varphi_k\}$ is infinite.*

Proof. It suffices to apply Theorem 3.6 for $l_n := pn$, $n \in \mathbb{N}$. Indeed, it holds

$$\sup_{k\in\mathbb{Z}} \tau\left(\varphi_{k+l_i}, \varphi_{k+l_j}\right) = \sup_{k\in\mathbb{Z}} \tau\left(\varphi_{k+l_i-l_j}, \varphi_k\right), \quad i,j \in \mathbb{N}.$$

□

Using Theorem 3.6 n-times, we also obtain the following result.

Corollary 3.8. *Let $(S_1,\tau_1),\dots,(S_n,\tau_n)$ be metric spaces and $\{\varphi_k^1\}_{k\in\mathbb{Z}},\dots,\{\varphi_k^n\}_{k\in\mathbb{Z}}$ be arbitrary sequences with values in $S_1,\dots,S_n$, respectively. The sequence $\{\psi_k\}_{k\in\mathbb{Z}}$, with values in $S_1\times\cdots\times S_n$ given by*

$$\psi_k = \left(\varphi_k^1,\dots,\varphi_k^n\right), \quad k \in \mathbb{Z},$$

is almost periodic if and only if all sequences $\{\varphi_k^1\},\dots,\{\varphi_k^n\}$ are almost periodic.

Definition 3.9. We say that a sequence $\{\varphi_k\}_{k\in\mathbb{Z}} \subseteq S$ is *asymptotically almost periodic* if, for every $\varepsilon > 0$, there exist $r(\varepsilon), R(\varepsilon) \in \mathbb{N}$ such that any set consisting of $r(\varepsilon)$ consecutive integers contains at least one number l satisfying

$$\tau\left(\varphi_{k+l}, \varphi_k\right) < \varepsilon, \quad k, k+l \geq R(\varepsilon).$$

Remark 3.10. Considering Theorem 3.5, we know that any limit periodic sequence is almost periodic. In addition, any almost periodic sequence is evidently asymptotically almost periodic. Note that, in Banach spaces, a sequence is asymptotically almost periodic if and only if it can be expressed as the sum of an almost periodic sequence and a sequence vanishing at infinity (see, e.g., [36, Chapter 5]).

4 Homogeneous linear difference systems over a field

In this section, we describe the studied systems in more details. Let $\mathcal{X} \subset Mat(F,m)$ be an arbitrarily given group. We recall that we will analyse homogeneous linear difference systems (1.1). Let $\mathcal{LP}(\mathcal{X})$ denotes the set of all systems (1.1) for which the sequence of matrices A_k is limit periodic. Analogously, the set of all almost periodic systems (1.1) will be denoted by $\mathcal{AP}(\mathcal{X})$. Especially, we can identify the sequence $\{A_k\}$ with the system in the form (1.1) which is determined by $\{A_k\}$. In $\mathcal{AP}(\mathcal{X})$, we introduce the metric

$$\sigma\left(\{A_k\},\{B_k\}\right) := \sup_{k\in\mathbb{Z}} \varrho\left(A_k, B_k\right), \quad \{A_k\},\{B_k\} \in \mathcal{AP}(\mathcal{X}).$$

Henceforth, the symbol $\mathcal{O}_\varepsilon^\sigma(\{A_k\})$ will denote the ε-neighbourhood of $\{A_k\}$ in $\mathcal{AP}(\mathcal{X})$.

Now we recall a definition from [35] which is used in the formulations of the below given Theorems 4.2 and 4.3 (for their proofs, see [35]). We point out that Theorems 4.2 and 4.3 are the basic motivation for our current research.

Definition 4.1. We say that $\mathcal{X}$ has *property P* if there exists $\zeta > 0$ and if, for all $\delta > 0$, there exists $l \in \mathbb{N}$ such that, for any vector $u \in F^m$ satisfying $\| u \| \geq 1$, one can find matrices $N_1, N_2, \dots, N_l \in \mathcal{X}$ with the property that

$$N_1 \in \mathcal{O}^{\varrho}_{\delta}(I), \quad N_i \in \mathcal{O}^{\varrho}_{\delta}(N_{i+1}),\ i \in \{1, \dots, l-1\}, \quad \| N_l \cdot u - u \| > \zeta.$$

Theorem 4.2. *Let $\mathcal{X}$ be bounded and have property P. For any $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$ which does not have any non-zero asymptotically almost periodic solution.*

Theorem 4.3. *Let $\mathcal{X}$ be bounded and have property P. For any $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$ which does not have any non-zero asymptotically almost periodic solution.*

In this paper, we intend to improve the above theorems. To show how the presented results improve Theorems 4.2 and 4.3, we need to reformulate Definition 4.1 for bounded groups applying the next two lemmas (which we will need later as well).

Lemma 4.4. *Let $p \in \mathbb{N}$ be given. The multiplication of p matrices is continuous in the Lipschitz sense on any bounded subset of $Mat(F, m)$.*

Proof. Let $K > 0$ be given. Since the addition and the multiplication have the Lipschitz property on the set of $f \in F$ satisfying $| f | < K$, the statement of the lemma is true. □

Lemma 4.5. *Let a bounded group $X \subseteq Mat(F, m)$ be given. There exists $L > 1$ such that*

$$M \cdot N^{-1}, N^{-1} \cdot M \in \mathcal{O}^{\varrho}_{aL}(I) \quad \text{if} \quad M, N \in X,\ M \in \mathcal{O}^{\varrho}_{a}(N). \tag{4.1}$$

Proof. We know that the inequality

$$\| M \| < K, \quad M \in X, \qquad \text{i.e.,} \qquad \left\| M^{-1} \right\| < K, \quad M \in X, \tag{4.2}$$

holds for some $K > 0$. The map $f \mapsto -f$, the multiplication, and the addition have the Lipschitz property on the set of all $f \in F$ satisfying $| f | < K$. In addition, for any $M \in X$, we have (see (4.2))

$$\det M < m!K^m, \quad \det M = \frac{1}{\det M^{-1}} > \frac{1}{m!K^m}.$$

Hence, the map

$$M \mapsto \frac{1}{\det M}, \quad M \in X,$$

has the Lipschitz property as well. Let a matrix $M \in X$ be given. If we use the expression

$$m^{-1}_{i,j} = \frac{M_{j,i}}{\det M}, \quad i, j \in \{1, \dots, m\},$$

where $m^{-1}_{i,j}$ are elements of $M^{-1} \in X$ and $M_{j,i}$ are the algebraic complements of the elements $m_{j,i}$ of M, then it is seen that the map $M \mapsto M^{-1}$ is continuous in the Lipschitz sense on X.

Evidently, Lemma 4.4 and the Lipschitz continuity of $M \mapsto M^{-1}$ on X imply the existence of $L > 1$ for which (4.1) is valid. □

Using Lemmas 4.4 and 4.5 for bounded $\mathcal{X}$ and for

$$N_1 = M_1,\ N_2 = M_2 \cdot M_1, \dots, N_l = M_l \cdots M_2 \cdot M_1,$$

we can rewrite Definition 4.1 as follows.

Definition 4.6. A bounded group $\mathcal{X} \subset Mat(F,m)$ has *property P* if there exists $\zeta > 0$ and if, for all $\delta > 0$, there exists $l \in \mathbb{N}$ such that, for any vector $u \in F^m$ satisfying $\| u \| \geq 1$, one can find matrices $M_1, M_2, \dots, M_l \in \mathcal{X}$ with the property that

$$M_i \in \mathcal{O}^{\varrho}_{\delta}(I),\ i \in \{1,2,\dots,l\}, \quad \| M_l \cdots M_2 \cdot M_1 \cdot u - u \| > \zeta.$$

To formulate the obtained results in a simple and consistent form, we introduce the following direct generalization of Definition 4.6.

Definition 4.7. Let a non-zero vector $u \in F^m$ be given. We say that $\mathcal{X}$ has *property P with respect to u* if there exists $\zeta > 0$ such that, for all $\delta > 0$, one can find matrices $M_1, M_2, \dots, M_l \in \mathcal{X}$ satisfying

$$M_i \in \mathcal{O}^{\varrho}_{\delta}(I),\ i \in \{1,2,\dots,l\}, \quad \| M_l \cdots M_2 \cdot M_1 \cdot u - u \| > \zeta.$$

Remark 4.8. Since a group with property P has property P with respect to any non-zero vector u (consider $\| f \odot u \| = | f | \cdot \| u \|,\ f \in F,\ u \in F^m$), we can refer to a lot of examples of matrix groups with property P mentioned in our previous paper [35]. In [35], there is also proved the following implication. If a complex transformable matrix group contains a matrix M satisfying $Mu \neq u$ for a vector $u \in \mathbb{C}^m$, then the group has property P with respect to u. Thus, concerning examples of groups having property P with respect to a given vector, we can also refer to our articles [13, 33], where (weakly) transformable groups are studied. Furthermore, we point out that any group, which contains a subgroup having property P with respect to a vector u, has property P with respect to u as well.

5 Results

Henceforth, we will assume that $\mathcal{X}$ is commutative. To prove the announced result (the below given Theorem 5.3), we use Lemmas 5.1 and 5.2.

Lemma 5.1. *Let $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrarily given. Let $\{\delta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a decreasing sequence satisfying*

$$\lim_{n\to\infty} \delta_n = 0 \tag{5.1}$$

and let $\{B^n_k\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ be periodic sequences for $n \in \mathbb{N}$ such that

$$B^n_k \in \mathcal{O}^{\varrho}_{\delta_n}(I), \quad k \in \mathbb{Z},\ n \in \mathbb{N}, \tag{5.2}$$

$$B^j_k = I \quad \text{or} \quad B^i_k = I, \quad k \in \mathbb{Z},\ i \neq j,\ i,j \in \mathbb{N}. \tag{5.3}$$

If one puts

$$B_k := A_k \cdot B^1_k \cdot B^2_k \cdots B^n_k \cdots, \quad k \in \mathbb{Z},$$

then $\{B_k\} \in \mathcal{LP}(\mathcal{X})$. In addition, if

$$\delta_1 < \frac{\varepsilon}{\sup\limits_{l\in\mathbb{Z}} \| A_l \|}, \tag{5.4}$$

then $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\})$.

Proof. Condition (5.3) means that, for any $k \in \mathbb{Z}$, there exists $i \in \mathbb{N}$ such that

$$B_k = A_k \cdot B_k^i. \tag{5.5}$$

Especially, the definition of $\{B_k\}_{k\in\mathbb{Z}}$ is correct and $B_k \in \mathcal{X}$, $k \in \mathbb{Z}$.

We show that $\{B_k\}$ is limit periodic. Since $\{A_k\}$ is limit periodic and $A_k \in \mathcal{X}$, $k \in \mathbb{Z}$, there exist periodic sequences $\{C_k^n\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ for $n \in \mathbb{N}$ with the property that

$$\| A_k - C_k^n \| < \frac{1}{n}, \quad k \in \mathbb{Z},\ n \in \mathbb{N}. \tag{5.6}$$

Let $\{B_k^n\}$ and $\{C_k^n\}$ have period $p_n \in \mathbb{N}$ and $q_n \in \mathbb{N}$ for $n \in \mathbb{N}$, respectively. The sequence $\{C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ has period $q_n \cdot p_1 \cdot p_2 \cdots p_n$; i.e., it is periodic for all $n \in \mathbb{N}$. It is valid that

$$\begin{aligned}
&\left\| B_k - C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \right\| \\
&\quad \le \left\| B_k - C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \cdots \right\| + \left\| C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \cdots - C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \right\| \\
&\quad \le \| A_k - C_k^n \| \cdot \left\| B_k^1 \cdot B_k^2 \cdots B_k^n \cdots \right\| + \left\| C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \right\| \cdot \left\| B_k^{n+1} \cdots B_k^{n+j} \cdots - I \right\|
\end{aligned}$$

and that

$$\begin{aligned}
\left\| C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \right\| &\le \| C_k^n \| \cdot \left\| B_k^1 \cdot B_k^2 \cdots B_k^n \right\| \\
&\le (\| A_k \| + \| C_k^n - A_k \|) \cdot \left\| B_k^1 \cdot B_k^2 \cdots B_k^n \right\|.
\end{aligned}$$

Hence (see (5.2), (5.3), (5.6)), we have

$$\left\| B_k - C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \right\| < \frac{1}{n}(m + \delta_1) + \left(\sup_{l\in\mathbb{Z}} \| A_l \| + \frac{1}{n} \right) (m + \delta_1)\, \delta_{n+1}$$

for all $k \in \mathbb{Z}$, $n \in \mathbb{N}$. Considering (5.1), we get that $\{B_k\}$ is the uniform limit of the sequence of periodic sequences $\{C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n\}$. Especially, $\{B_k\} \in \mathcal{LP}(\mathcal{X})$.

Let (5.4) be true. We have to prove that $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$, i.e.,

$$\sup_{k\in\mathbb{Z}} \| A_k - B_k \| < \varepsilon. \tag{5.7}$$

Since

$$B_k^n \in \mathcal{O}_{\delta_1}^{\varrho}(I), \quad k \in \mathbb{Z},\ n \in \mathbb{N},$$

considering (5.5), we have

$$\| A_k - B_k \| \le \| A_k \| \cdot \left\| I - B_k^i \right\| \le \delta_1 \sup_{l\in\mathbb{Z}} \| A_l \|$$

for some $i \in \mathbb{N}$ and for all $k \in \mathbb{Z}$. Thus (see (5.4)), we obtain (5.7). □

Lemma 5.2. *If for any $\delta > 0$ and $K > 0$, there exist matrices $M_1, M_2, \ldots, M_l \in \mathcal{X}$ such that*

$$M_i \in \mathcal{O}_\delta^\varrho(I),\ i \in \{1, 2, \ldots, l\}, \quad \| M_l \cdots M_2 \cdot M_1 \| > K,$$

then, for any $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$ whose fundamental matrix is not almost periodic.

Proof. We can assume that all solutions of $\{A_k\}$ are almost periodic. Especially (consider Corollary 3.8), for any $\vartheta > 0$, there exist infinitely many positive integers p with the property that

$$\| A_{p-1} \cdots A_1 \cdot A_0 - I \| < \vartheta. \tag{5.8}$$

Let $\{\delta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a decreasing sequence satisfying (5.1) and (5.4). For δ_n and $K_n := n$, $n \in \mathbb{N}$, we consider matrices

$$\begin{gathered} M_1^1, M_2^1, \dots, M_{l_1}^1 \in \mathcal{X}, \\ M_1^2, M_2^2, \dots, M_{l_2}^2 \in \mathcal{X}, \\ \vdots \\ M_1^j, M_2^j, \dots, M_{l_j}^j \in \mathcal{X}, \\ \vdots \end{gathered}$$

such that

$$M_i^j \in \mathcal{O}_{\delta_j}^{\varrho}(I), \quad i \in \{1, 2, \dots, l_j\}, \ j \in \mathbb{N}, \tag{5.9}$$

and

$$\left\| M_{l_j}^j \cdots M_2^j \cdot M_1^j \right\| > K_j = j, \quad j \in \mathbb{N}. \tag{5.10}$$

Let a sequence of positive numbers ϑ_n for $n \in \mathbb{N}$ be given.

Let us consider $p_1^1, p_2^1 \in \mathbb{N}$ such that $p_2^1 - p_1^1 > 2l_1$ and that (see (5.8))

$$\left\| A_{p_2^1 - 1} \cdots A_1 \cdot A_0 - I \right\| < \vartheta_1. \tag{5.11}$$

In addition, let p_1^1 and p_2^1 be even (consider Corollary 3.7). We define the periodic sequence $\{B_k^1\}_{k\in\mathbb{Z}}$ with period p_2^1 by values

$$B_0^1 := I, \ B_1^1 := I, \dots, B_{p_1^1 - 2}^1 := I,$$

$$B_{p_1^1 - 1}^1 := I, \ B_{p_1^1}^1 := I, \ B_{p_1^1 + 1}^1 := M_1^1, \ B_{p_1^1 + 2}^1 := I, \ B_{p_1^1 + 3}^1 := M_2^1, \ B_{p_1^1 + 4}^1 := I,$$

$$\vdots$$

$$\begin{gathered} B_{p_1^1 + 2l_1 - 3}^1 := M_{l_1 - 1}^1, \ B_{p_1^1 + 2l_1 - 2}^1 := I, \ B_{p_1^1 + 2l_1 - 1}^1 := M_{l_1}^1, \\ B_{p_1^1 + 2l_1}^1 := I, \ B_{p_1^1 + 2l_1 + 1}^1 := I, \ B_{p_1^1 + 2l_1 + 2}^1 := I, \end{gathered}$$

$$\vdots$$

$$B_{p_2^1 - 1}^1 := I.$$

We put

$$\tilde{B}_k^1 := A_k \cdot B_k^1, \quad k \in \mathbb{Z}.$$

We have

$$\left\| \tilde{B}_{p_2^1 - 1}^1 \cdots \tilde{B}_1^1 \cdot \tilde{B}_0^1 \right\| = \left\| M_{l_1}^1 \cdots M_2^1 \cdot M_1^1 \cdot A_{p_2^1 - 1} \cdots A_1 \cdot A_0 \right\|.$$

Again, we can assume that, for any $\vartheta > 0$, there exist infinitely many positive integers p with the property that

$$\left\| \tilde{B}^1_{p-1} \cdots \tilde{B}^1_1 \cdot \tilde{B}^1_0 - I \right\| < \vartheta. \tag{5.12}$$

Otherwise, we obtain the system $\{B_k\} \equiv \{\tilde{B}^1_k\}$ with a non-almost periodic solution. Indeed, it suffices to consider Lemma 5.1 for $B^{n+1}_k = I$, $k \in \mathbb{Z}$, $n \in \mathbb{N}$.

Analogously, let us consider $p^2_1, p^2_2 \in \mathbb{N}$ satisfying $p^2_2 - 4l_2 > p^2_1 > p^1_2$ and (see (5.12))

$$\left\| \tilde{B}^1_{p^2_2-1} \cdots \tilde{B}^1_1 \cdot \tilde{B}^1_0 - I \right\| < \vartheta_2. \tag{5.13}$$

Let $p^2_1, p^2_2 \in 4\mathbb{N}$ (see Corollary 3.7). We define the periodic sequence $\{B^2_k\}_{k\in\mathbb{Z}}$ with period p^2_2 by values

$$B^2_0 := I,\ B^2_1 := I, \ldots, B^2_{p^2_1-1} := I,$$

$$B^2_{p^2_1} := I,\ B^2_{p^2_1+1} := I,\ B^2_{p^2_1+2} := M^2_1,\ B^2_{p^2_1+3} := I,$$

$$B^2_{p^2_1+4} := I,\ B^2_{p^2_1+5} := I,\ B^2_{p^2_1+6} := M^2_2,\ B^2_{p^2_1+7} := I,$$

$$\vdots$$

$$B^2_{p^2_1+4l_2-4} := I,\ B^2_{p^2_1+4l_2-3} := I,\ B^2_{p^2_1+4l_2-2} := M^2_{l_2},\ B^2_{p^2_1+4l_2-1} := I,$$

$$B^2_{p^2_1+4l_2} := I,\ B^2_{p^2_1+4l_2+1} := I,\ B^2_{p^2_1+4l_2+2} := I,\ B^2_{p^2_1+4l_2+3} := I,$$

$$\vdots$$

$$B^2_{p^2_2-1} := I.$$

For

$$\tilde{B}^2_k := A_k \cdot B^1_k \cdot B^2_k, \quad k \in \mathbb{Z},$$

it holds

$$\left\| \tilde{B}^2_{p^2_2-1} \cdots \tilde{B}^2_1 \cdot \tilde{B}^2_0 \right\| = \left\| M^2_{l_2} \cdots M^2_2 \cdot M^2_1 \cdot \tilde{B}^1_{p^2_2-1} \cdots \tilde{B}^1_1 \cdot \tilde{B}^1_0 \right\|.$$

Especially, for all $k \in \mathbb{Z}$, there exists $i \in \{1, 2\}$ such that $\tilde{B}^2_k := A_k \cdot B^i_k$.

We continue in the same manner. Let us assume that all obtained systems $\{\tilde{B}^j_k\}_{k\in\mathbb{Z}}$ have only almost periodic solutions. Thus, for every $\vartheta > 0$ and $j \in \mathbb{N}$, one can find infinitely many $p \in \mathbb{N}$ such that

$$\left\| \tilde{B}^j_{p-1} \cdots \tilde{B}^j_1 \cdot \tilde{B}^j_0 - I \right\| < \vartheta.$$

In the n-th step, we consider $p^n_1, p^n_2 \in 2^n\mathbb{N}$ such that $p^n_2 - 2^n l_n > p^n_1 > p^{n-1}_2$ and

$$\left\| \tilde{B}^{n-1}_{p^n_2-1} \cdots \tilde{B}^{n-1}_1 \cdot \tilde{B}^{n-1}_0 - I \right\| < \vartheta_n. \tag{5.14}$$

We define the periodic sequence $\{B^n_k\}_{k\in\mathbb{Z}}$ with period p^n_2 by values

$$B^n_0 := I,\ B^n_1 := I, \ldots, B^n_{p^n_1-1} := I,$$

$$B^n_{p^n_1} := I,\, B^n_{p^n_1+1} := I, \dots, B^n_{p^n_1+2^{n-1}-1} := I,$$
$$B^n_{p^n_1+2^{n-1}} := M^n_1,\ B^n_{p^n_1+2^{n-1}+1} := I,\ \dots, B^n_{p^n_1+2^n-1} := I,$$

$$B^n_{p^n_1+2^n} := I,\, B^n_{p^n_1+2^n+1} := I, \dots, B^n_{p^n_1+2^n+2^{n-1}-1} := I,$$
$$B^n_{p^n_1+2^n+2^{n-1}} := M^n_2,\ B^n_{p^n_1+2^n+2^{n-1}+1} := I, \dots, B^n_{p^n_1+2\cdot 2^n-1} := I,$$
$$\vdots$$
$$B^n_{p^n_1+(l_n-1)2^n} := I,\, B^n_{p^n_1+(l_n-1)2^n+1} := I, \dots, B^n_{p^n_1+(l_n-1)2^n+2^{n-1}-1} := I,$$
$$B^n_{p^n_1+(l_n-1)2^n+2^{n-1}} := M^n_{l_n},\ B^n_{p^n_1+(l_n-1)2^n+2^{n-1}+1} := I, \dots, B^n_{p^n_1+l_n2^n-1} := I,$$

$$B^n_{p^n_1+l_n2^n} := I,\, B^n_{p^n_1+l_n2^n+1} := I, \dots, B^n_{p^n_1+l_n2^n+2^{n-1}-1} := I,$$
$$B^n_{p^n_1+l_n2^n+2^{n-1}} := I,\ B^n_{p^n_1+l_n2^n+2^{n-1}+1} := I, \dots, B^n_{p^n_1+(l_n+1)2^n-1} := I,$$
$$\vdots$$
$$B^n_{p^n_2-1} := I.$$

If we put
$$\tilde{B}^n_k := A_k \cdot B^1_k \cdot B^2_k \cdots B^n_k, \quad k \in \mathbb{Z},$$
then
$$\left\| \tilde{B}^n_{p^n_2-1} \cdots \tilde{B}^n_1 \cdot \tilde{B}^n_0 \right\| = \left\| M^n_{l_n} \cdots M^n_2 \cdot M^n_1 \cdot \tilde{B}^{n-1}_{p^n_2-1} \cdots \tilde{B}^{n-1}_1 \cdot \tilde{B}^{n-1}_0 \right\|. \tag{5.15}$$

Finally, we put
$$B_k := A_k \cdot B^1_k \cdot B^2_k \cdots B^n_k \cdots, \quad k \in \mathbb{Z}.$$

From the construction, we obtain that, for any $k \in \mathbb{Z}$, there exists $i \in \mathbb{N}$ such that $B_k = A_k \cdot B^i_k$. It means that (5.3) is satisfied. Since (5.2) follows from the construction and from (5.9), we can use Lemma 5.1 which guarantees that $\{B_k\} \in \mathcal{O}^\sigma_\varepsilon(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$. It remains to prove that the fundamental matrix of $\{B_k\}$ is not almost periodic. On contrary, let us assume its almost periodicity. Then, the fundamental matrix is bounded (see Remark 3.4); i.e., there exists $K_0 > 0$ with the property that
$$\| B_k \cdots B_1 \cdot B_0 \| < K_0, \quad k \in \mathbb{N}. \tag{5.16}$$

Let us choose $n \in \mathbb{N}$ for which $n \geq K_0 + 1$. We repeat that the multiplication of matrices is continuous (see also Lemma 4.4). Hence, for given matrix
$$M^n_1 \cdot M^n_2 \cdots M^n_{l_n} = M^n_{l_n} \cdots M^n_2 \cdot M^n_1 \in \mathcal{X},$$
there exists $\theta_n > 0$ such that
$$\| M^n_{l_n} \cdots M^n_2 \cdot M^n_1 \| - 1 < \| M^n_{l_n} \cdots M^n_2 \cdot M^n_1 \cdot C \|, \quad C \in \mathcal{O}^\varrho_{\theta_n}(I). \tag{5.17}$$

We can assume that $\vartheta_n = \theta_n$ in (5.14) (see also (5.11), (5.13)). We construct sequences $\{B^j_k\}$ in such a way that
$$B^j_0 = I,\ B^j_1 = I, \dots, B^j_{p^n_2-1} = I, \quad j > n,\ j,n \in \mathbb{N}.$$

Indeed, $p_1^{j+1} > p_2^j > p_1^j$, $j \in \mathbb{N}$. Thus, (5.10), (5.14), (5.15), and (5.17) imply

$$\begin{aligned} \left\| B_{p_2^n - 1} \cdots B_1 \cdot B_0 \right\| &= \left\| \tilde{B}^n_{p_2^n - 1} \cdots \tilde{B}^n_1 \cdot \tilde{B}^n_0 \right\| \\ &= \left\| M^n_{l_n} \cdots M^n_2 \cdot M^n_1 \cdot \tilde{B}^{n-1}_{p_2^n - 1} \cdots \tilde{B}^{n-1}_1 \cdot \tilde{B}^{n-1}_0 \right\| \\ &> \left\| M^n_{l_n} \cdots M^n_2 \cdot M^n_1 \right\| - 1 > n - 1 \geq K_0. \end{aligned} \tag{5.18}$$

This contradiction (cf. (5.16) and (5.18)) completes the proof. □

Theorem 5.3. *Let $\mathcal{X}$ have property P with respect to a vector u. For any $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}^\sigma_\varepsilon(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$ whose fundamental matrix is not almost periodic.*

Proof. Let us consider the solution $\{x^0_k\}_{k\in\mathbb{Z}}$ of the Cauchy problem

$$x_{k+1} = A_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

If $\{x^0_k\}$ is not almost periodic, then the statement of the theorem is true for $B_k := A_k$, $k \in \mathbb{Z}$. Hence, we assume that $\{x^0_k\}$ is almost periodic.

We put

$$\delta_n := \frac{1}{n+1} \cdot \frac{\varepsilon}{\sup\limits_{l\in\mathbb{Z}} \| A_l \|}, \quad n \in \mathbb{N}. \tag{5.19}$$

We know that there exist $\zeta > 0$ and matrices

$$\begin{gathered} M^1_1, M^1_2, \ldots, M^1_{l_1} \in \mathcal{X}, \\ M^2_1, M^2_2, \ldots, M^2_{l_2} \in \mathcal{X}, \\ \vdots \\ M^j_1, M^j_2, \ldots, M^j_{l_j} \in \mathcal{X}, \\ \vdots \end{gathered}$$

such that

$$M^j_i \in \mathcal{O}^\varrho_{\delta_j}(I), \ i \in \{1, \ldots, l_j\}, \tag{5.20}$$

$$\left\| M^j_{l_j} \cdots M^j_2 \cdot M^j_1 \cdot u - u \right\| > \zeta \tag{5.21}$$

for all $j \in \mathbb{N}$. Of course, we can consider l_j such that

$$l_j \geq \cdots \geq l_2 \geq l_1 \geq 2. \tag{5.22}$$

Denote

$$K_j := \left\| M^j_{l_j} \cdots M^j_2 \cdot M^j_1 \right\|, \quad j \in \mathbb{N}. \tag{5.23}$$

For

$$\vartheta_j := \frac{\zeta}{2\,(K_j + 2)}, \quad j \in \mathbb{N}, \tag{5.24}$$

we have

$$\| M \cdot v - w \| > \frac{\zeta}{2} \quad \text{if} \quad \| M \cdot u - u \| > \zeta,\ M \in \mathcal{O}^\varrho_{K_j+1}(O) \cap \mathcal{X},\ v, w \in \mathcal{O}^\varrho_{\vartheta_j}(u). \tag{5.25}$$

Indeed, for considered $u, v, w \in F^m$ and $M \in \mathcal{X}$, it holds (see (5.24))

$$\begin{aligned} \| M \cdot u - u \| &\leq \| M \cdot u - M \cdot v \| + \| M \cdot v - w \| + \| w - u \| \\ &< \left(K_j + 1\right) \| u - v \| + \| w - u \| + \| M \cdot v - w \| < \frac{\zeta}{2} + \| M \cdot v - w \| . \end{aligned}$$

The almost periodicity of $\{x_k^0\}$ (see Corollary 3.7) implies that there exists an even positive integer $j_{(1,0)}$ such that

$$\left\| x_0^0 - x_{j_{(1,0)}}^0 \right\| = \left\| u - x_{j_{(1,0)}}^0 \right\| < \frac{\vartheta_1}{2}. \tag{5.26}$$

Let us define a periodic sequence $\{B_k^1\}$ with period $j(1,0) + r_1$, where $r_1 := 2l_1$. If

$$\left\| x_{j_{(1,0)}}^0 - x_{j_{(1,0)}+r_1}^0 \right\| \geq \frac{\vartheta_1}{2}, \tag{5.27}$$

then we put $B_k^1 := I$, $k \in \mathbb{Z}$; and if

$$\left\| x_{j_{(1,0)}}^0 - x_{j_{(1,0)}+r_1}^0 \right\| < \frac{\vartheta_1}{2}, \tag{5.28}$$

then

$$\begin{gathered} B_0^1 := I,\ B_1^1 := I, \dots, B_{j_{(1,0)}-1}^1 := I, \\ B_{j_{(1,0)}}^1 := I,\ B_{j_{(1,0)}+1}^1 := M_1^1,\ B_{j_{(1,0)}+2}^1 := I,\ B_{j_{(1,0)}+3}^1 := M_2^1, \\ \vdots \\ B_{j_{(1,0)}+2l_1-4}^1 := I,\ B_{j_{(1,0)}+2l_1-3}^1 := M_{l_1-1}^1,\ B_{j_{(1,0)}+2l_1-2}^1 := I,\ B_{j_{(1,0)}+2l_1-1}^1 := M_{l_1}^1. \end{gathered}$$

For $\tilde{B}_k^1 := A_k \cdot B_k^1$, $k \in \mathbb{Z}$, we consider the solution $\{x_k^1\}_{k \in \mathbb{Z}}$ of the initial problem

$$x_{k+1} = \tilde{B}_k^1 \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

Lemma 5.1 gives that $\{\tilde{B}_k^1\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$. In the case when $\{x_k^1\}$ is not almost periodic, we can put $B_k := \tilde{B}_k^1$, $k \in \mathbb{Z}$. Thus, we have to consider the almost periodicity of $\{x_k^1\}$. Especially (see Corollary 3.7), there exist infinitely many numbers $j \in 4\mathbb{N}$ with the property that

$$\left\| x_0^1 - x_j^1 \right\| = \left\| u - x_j^1 \right\| < \frac{\vartheta_2}{2}. \tag{5.29}$$

Let us consider an integer $j_{(1,1)} \in 4\mathbb{N}$ satisfying (5.29) and the inequality

$$j_{(1,1)} \geq j_{(1,0)} + r_1. \tag{5.30}$$

For $r_2 := 8l_1l_2$, we define a sequence $\{B_k^{(1,2)}\}_{k \in \mathbb{Z}}$ with period $j_{(1,1)} + r_2$. We put $B_k^{(1,2)} := I$ for all $k \in \mathbb{Z}$ if

$$\left\| x_{j_{(1,1)}}^1 - x_{j_{(1,1)}+r_2}^1 \right\| \geq \frac{\vartheta_2}{2}. \tag{5.31}$$

In the second case, when (5.31) is not valid, we define

$$\begin{gathered} B_0^{(1,2)} := I,\ B_1^{(1,2)} := I,\ \dots, B_{j_{(1,1)}-1}^{(1,2)} := I, \\ B_{j_{(1,1)}}^{(1,2)} := I,\ B_{j_{(1,1)}+1}^{(1,2)} := I,\ B_{j_{(1,1)}+2}^{(1,2)} := M_1^2,\ B_{j_{(1,1)}+3}^{(1,2)} := I, \end{gathered}$$

$$B^{(1,2)}_{j_{(1,1)}+4} := I,\ B^{(1,2)}_{j_{(1,1)}+5} := I,\ B^{(1,2)}_{j_{(1,1)}+6} := M^2_2,\ B^{(1,2)}_{j_{(1,1)}+7} := I,$$
$$\vdots$$
$$B^{(1,2)}_{j_{(1,1)}+4l_2-4} := I,\ B^{(1,2)}_{j_{(1,1)}+4l_2-3} := I,\ B^{(1,2)}_{j_{(1,1)}+4l_2-2} := M^2_{l_2},\ B^{(1,2)}_{j_{(1,1)}+4l_2-1} := I,$$
$$B^{(1,2)}_{j_{(1,1)}+4l_2} := I,\ B^{(1,2)}_{j_{(1,1)}+4l_2+1} := I,\ B^{(1,2)}_{j_{(1,1)}+4l_2+2} := I,\ B^{(1,2)}_{j_{(1,1)}+4l_2+3} := I,$$
$$\vdots$$
$$B^{(1,2)}_{j_{(1,1)}+r_2-1} := I.$$

For $\tilde{B}^{(1,2)}_k := A_k \cdot B^1_k \cdot B^{(1,2)}_k$, $k \in \mathbb{Z}$, we consider the solution $\{x^{(1,2)}_k\}_{k\in\mathbb{Z}}$ of the initial problem

$$x_{k+1} = \tilde{B}^{(1,2)}_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

Again, we can assume that $\{x^{(1,2)}_k\}_{k\in\mathbb{Z}}$ is almost periodic. Let an integer $j_{(2,1)} \in 8\mathbb{N}$ have the properties that

$$\left\| x^{(1,2)}_0 - x^{(1,2)}_{j_{(2,1)}} \right\| = \left\| u - x^{(1,2)}_{j_{(2,1)}} \right\| < \frac{\vartheta_2}{2} \tag{5.32}$$

and that

$$j_{(2,1)} \geq j_{(1,1)} + r_2. \tag{5.33}$$

We define a periodic sequence $\{B^{(2,2)}_k\}_{k\in\mathbb{Z}}$ with period $j_{(2,1)}(r_2 - r_1)$. If

$$\left\| x^{(1,2)}_{j_{(2,1)}} - x^{(1,2)}_{j_{(2,1)}+r_2-r_1} \right\| \geq \frac{\vartheta_2}{2}, \tag{5.34}$$

we put $B^{(2,2)}_k := I$ for all $k \in \mathbb{Z}$; and, in the other case, we define

$$B^{(2,2)}_0 := I,\ B^{(2,2)}_1 := I, \dots, B^{(2,2)}_{j_{(2,1)}-1} := I,$$
$$B^{(2,2)}_{j_{(2,1)}} := I,\ B^{(2,2)}_{j_{(2,1)}+1} := I,\ B^{(2,2)}_{j_{(2,1)}+2} := I,\ B^{(2,2)}_{j_{(2,1)}+3} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+4} := M^2_1,\ B^{(2,2)}_{j_{(2,1)}+5} := I,\ B^{(2,2)}_{j_{(2,1)}+6} := I,\ B^{(2,2)}_{j_{(2,1)}+7} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+8} := I,\ B^{(2,2)}_{j_{(2,1)}+9} := I,\ B^{(2,2)}_{j_{(2,1)}+10} := I,\ B^{(2,2)}_{j_{(2,1)}+11} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+12} := M^2_2,\ B^{(2,2)}_{j_{(2,1)}+13} := I,\ B^{(2,2)}_{j_{(2,1)}+14} := I,\ B^{(2,2)}_{j_{(2,1)}+15} := I,$$
$$\vdots$$
$$B^{(2,2)}_{j_{(2,1)}+8l_2-8} := I,\ B^{(2,2)}_{j_{(2,1)}+8l_2-7} := I,\ B^{(2,2)}_{j_{(2,1)}+8l_2-6} := I,\ B^{(2,2)}_{j_{(2,1)}+8l_2-5} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+8l_2-4} := M^2_{l_2},\ B^{(2,2)}_{j_{(2,1)}+8l_2-3} := I,\ B^{(2,2)}_{j_{(2,1)}+8l_2-2} := M^2_{l_2},\ B^{(2,2)}_{j_{(2,1)}+8l_2-1} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+8l_2} := I,\ B^{(2,2)}_{j_{(2,1)}+8l_2+1} := I,\ B^{(2,2)}_{j_{(2,1)}+8l_2+2} := I,\ B^{(2,2)}_{j_{(2,1)}+8l_2+3} := I,$$
$$B^{(2,2)}_{j_{(2,1)}+8l_2+4} := I,\ B^{(2,2)}_{j_{(2,1)}+8l_2+5} := I,\ B^{(2,2)}_{j_{(2,1)}+8l_2+6} := I,\ B^{(2,2)}_{j_{(2,1)}+8l_2+7} := I,$$
$$\vdots$$

$$B^{(2,2)}_{j_{(2,1)}+r_2-r_1-1} := I, \dots, B^{(2,2)}_{j_{(2,1)}(r_2-r_1)-1} := I.$$

Finally, in the second step, we consider the periodic sequence of

$$B_k^2 := B_k^{(1,2)} \cdot B_k^{(2,2)}, \quad k \in \mathbb{Z}. \tag{5.35}$$

Note that its period is $[j_{(1,1)} + r_2][j_{(2,1)}(r_2 - r_1)]$. Consequently, we consider

$$\tilde{B}_k^2 := A_k \cdot B_k^1 \cdot B_k^2, \quad k \in \mathbb{Z}, \tag{5.36}$$

and the solution $\{x_k^2\}_{k\in\mathbb{Z}}$ of

$$x_{k+1} = \tilde{B}_k^2 \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

In the case when $\{x_k^2\}$ is not almost periodic, we can put $B_k := \tilde{B}_k^2$ for $k \in \mathbb{Z}$ and use Lemma 5.1 for $B_k^{j+2} = I$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$ (see also (5.36)). Thus, we have to assume that $\{x_k^2\}$ is almost periodic.

We continue in the same manner. Before the n-th step, we define

$$\tilde{B}_k^{n-1} := A_k \cdot B_k^1 \cdot B_k^2 \cdots B_k^{n-1}, \quad k \in \mathbb{Z}.$$

Let $\{\tilde{B}_k^{n-1}\}_{k\in\mathbb{Z}}$ have period q_{n-1}, e.g., let

$$\begin{aligned} q_{n-1} :=& [j_{(1,0)} + r_1][j_{(1,1)} + r_2][j_{(2,1)}(r_2 - r_1)] \times \cdots \\ & \cdots \times [j_{(1,n-2)} + r_{n-1}][j_{(2,n-2)}(r_{n-1} - r_1)] \cdots [j_{(n-1,n-2)}(r_{n-1} - r_{n-2})]. \end{aligned}$$

Consider the solution $\{x_k^{n-1}\}$ of

$$x_{k+1} = \tilde{B}_k^{n-1} \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

Again, we consider that the sequence $\{x_k^{n-1}\}$ is almost periodic. Otherwise, we can put $B_k := \tilde{B}_k^{n-1}$, $k \in \mathbb{Z}$. Especially, for all $p \in \mathbb{N}$, there exist infinitely many numbers $j \in p\mathbb{N}$ with the property that

$$\left\| x_0^{n-1} - x_j^{n-1} \right\| = \left\| u - x_j^{n-1} \right\| < \frac{\vartheta_n}{2}. \tag{5.37}$$

Denote

$$p_n := 2^{1+\sum_{i=1}^{n-1} i}, \quad n \geq 2,\, n \in \mathbb{N}, \tag{5.38}$$

$$r_n := 2p_n l_1 l_2 \cdots l_n, \quad n \geq 2,\, n \in \mathbb{N}. \tag{5.39}$$

Let us consider an integer $j_{(1,n-1)} \in p_n\mathbb{N}$ satisfying (5.37) and

$$j_{(1,n-1)} \geq q_{n-1}. \tag{5.40}$$

We define $\{B_k^{(1,n)}\}_{k\in\mathbb{Z}}$ with period $j_{(1,n-1)} + r_n$. If

$$\left\| x^{n-1}_{j_{(1,n-1)}} - x^{n-1}_{j_{(1,n-1)}+r_n} \right\| \geq \frac{\vartheta_n}{2}, \tag{5.41}$$

we put $B_k^{(1,n)} := I$, $k \in \mathbb{Z}$. In the other case, we put

$$B_0^{(1,n)} := I,\ B_1^{(1,n)} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}-1} := I,$$

$$B^{(1,n)}_{j_{(1,n-1)}} := I,\ B^{(1,n)}_{j_{(1,n-1)}+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+\frac{p_n}{2}-1} := I,$$

$$B^{(1,n)}_{j_{(1,n-1)}+\frac{p_n}{2}} := M^n_1,\ B^{(1,n)}_{j_{(1,n-1)}+\frac{p_n}{2}+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+p_n-1} := I,$$

$$B^{(1,n)}_{j_{(1,n-1)}+p_n} := I,\ B^{(1,n)}_{j_{(1,n-1)}+p_n+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+p_n+\frac{p_n}{2}-1} := I,$$

$$B^{(1,n)}_{j_{(1,n-1)}+p_n+\frac{p_n}{2}} := M^n_2,\ B^{(1,n)}_{j_{(1,n-1)}+p_n+\frac{p_n}{2}+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+2p_n-1} I,$$

$$\vdots$$

$$B^{(1,n)}_{j_{(1,n-1)}+(l_n-1)p_n} := I,\ B^{(1,n)}_{j_{(1,n-1)}+(l_n-1)p_n+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+(l_n-1)p_n+\frac{p_n}{2}-1} := I,$$

$$B^{(1,n)}_{j_{(1,n-1)}+(l_n-1)p_n+\frac{p_n}{2}} := M^n_{l_n},\ B^{(1,n)}_{j_{(1,n-1)}+(l_n-1)p_n+\frac{p_n}{2}+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+l_n p_n-1} := I,$$

$$B^{(1,n)}_{j_{(1,n-1)}+l_n p_n} := I,\ B^{(1,n)}_{j_{(1,n-1)}+l_n p_n+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+l_n p_n+\frac{p_n}{2}-1} := I,$$

$$B^{(1,n)}_{j_{(1,n-1)}+l_n p_n+\frac{p_n}{2}} := I,\ B^{(1,n)}_{j_{(1,n-1)}+l_n p_n+\frac{p_n}{2}+1} := I, \dots, B^{(1,n)}_{j_{(1,n-1)}+(l_n+1)p_n-1} := I,$$

$$\vdots$$

$$B^{(1,n)}_{j_{(1,n-1)}+r_n-1} := I.$$

For
$$\tilde{B}^{(1,n)}_k := \tilde{B}^{n-1}_k \cdot B^{(1,n)}_k = A_k \cdot B^1_k \cdot B^2_k \cdots B^{n-1}_k \cdot B^{(1,n)}_k, \quad k \in \mathbb{Z},$$
we consider the solution $\{x^{(1,n)}_k\}$ of

$$x_{k+1} = \tilde{B}^{(1,n)}_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

Again, we assume that $\{x^{(1,n)}_k\}$ is almost periodic. Let a number $j_{(2,n-1)} \in 2p_n\mathbb{N}$ have the properties that
$$\left\| x^{(1,n)}_0 - x^{(1,n)}_{j_{(2,n-1)}} \right\| = \left\| u - x^{(1,n)}_{j_{(2,n-1)}} \right\| < \frac{\vartheta_n}{2} \tag{5.42}$$
and
$$j_{(2,n-1)} \ge j_{(1,n-1)} + r_n. \tag{5.43}$$

We define the following periodic sequence $\{B^{(2,n)}_k\}_{k\in\mathbb{Z}}$ with period $j_{(2,n-1)}(r_n - r_1)$. If

$$\left\| x^{(1,n)}_{j_{(2,n-1)}} - x^{(1,n)}_{j_{(2,n-1)}+r_n-r_1} \right\| \ge \frac{\vartheta_n}{2}, \tag{5.44}$$

then $B^{(2,n)}_k := I$, $k \in \mathbb{Z}$. In the other case, we put

$$B^{(2,n)}_0 := I,\ B^{(2,n)}_1 := I, \dots, B^{(2,n)}_{j_{(2,n-1)}-1} := I,$$

$$B^{(2,n)}_{j_{(2,n-1)}} := I,\ B^{(2,n)}_{j_{(2,n-1)}+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+p_n-1} := I,$$

$$B^{(2,n)}_{j_{(2,n-1)}+p_n} := M^n_1,\ B^{(2,n)}_{j_{(2,n-1)}+p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+2p_n-1} := I,$$

$$B^{(2,n)}_{j_{(2,n-1)}+2p_n} := I,\ B^{(2,n)}_{j_{(2,n-1)}+2p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+2p_n+p_n-1} := I,$$

$$B^{(2,n)}_{j_{(2,n-1)}+2p_n+p_n} := M^n_2,\ B^{(2,n)}_{j_{(2,n-1)}+2p_n+p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+4p_n-1} := I,$$

$$\vdots$$

$$B^{(2,n)}_{j_{(2,n-1)}+(l_n-1)2p_n} := I,\ B^{(2,n)}_{j_{(2,n-1)}+(l_n-1)2p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+(l_n-1)2p_n+p_n-1} := I,$$

$$B^{(2,n)}_{j_{(2,n-1)}+(l_n-1)2p_n+p_n} := M^n_{l_n},\ B^{(2,n)}_{j_{(2,n-1)}+(l_n-1)2p_n+p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+2l_np_n-1} := I,$$

$$B^{(2,n)}_{j_{(2,n-1)}+2l_np_n} := I,\ B^{(2,n)}_{j_{(2,n-1)}+2l_np_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+2l_np_n+p_n-1} := I,$$

$$B^{(2,n)}_{j_{(2,n-1)}+2l_np_n+p_n} := I,\ B^{(2,n)}_{j_{(2,n-1)}+2l_np_n+p_n+1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}+2(l_n+1)p_n-1} := I,$$

$$\vdots$$

$$B^{(2,n)}_{j_{(2,n-1)}+r_n-r_1-1} := I, \dots, B^{(2,n)}_{j_{(2,n-1)}(r_n-r_1)-1} := I.$$

We continue in the n-th step. We define

$$\tilde{B}^{(n-1,n)}_k := \tilde{B}^{n-1}_k \cdot B^{(1,n)}_k \cdot B^{(2,n)}_k \cdots B^{(n-1,n)}_k, \quad k \in \mathbb{Z}.$$

We consider the solution $\{x^{(n-1,n)}_k\}_{k\in\mathbb{Z}}$ of

$$x_{k+1} = \tilde{B}^{(n-1,n)}_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u.$$

Again, we have to assume that $\{x^{(n-1,n)}_k\}$ is almost periodic. Let $j_{(n,n-1)} \in 2^{n-1}p_n\mathbb{N}$ satisfy

$$\left\| x^{(n-1,n)}_0 - x^{(n-1,n)}_{j_{(n,n-1)}} \right\| = \left\| u - x^{(n-1,n)}_{j_{(n,n-1)}} \right\| < \frac{\vartheta_n}{2} \tag{5.45}$$

and

$$j_{(n,n-1)} \geq j_{(n-1,n-1)}(r_n - r_{n-2}). \tag{5.46}$$

We define a periodic sequence $\{B^{(n,n)}_k\}_{k\in\mathbb{Z}}$ with period $j_{(n,n-1)}(r_n - r_{n-1})$. If

$$\left\| x^{(n-1,n)}_{j_{(n,n-1)}} - x^{(n-1,n)}_{j_{(n,n-1)}+r_n-r_{n-1}} \right\| \geq \frac{\vartheta_n}{2}, \tag{5.47}$$

we put $B^{(n,n)}_k := I$, $k \in \mathbb{Z}$. If inequality (5.47) is not valid, we put

$$B^{(n,n)}_0 := I,\ B^{(n,n)}_1 := I, \dots, B^{(n,n)}_{j_{(n,n-1)}-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}} := I,\ B^{(n,n)}_{j_{(n,n-1)}+1} := I, \dots, B^{(n,n)}_{j_{(n,n-1)}+2^{n-2}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+2^{n-2}p_n} := M^n_1,\ B^{(n,n)}_{j_{(n,n-1)}+2^{n-2}p_n+1} := I, \dots, B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n} := I,\ B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n+1} := I, \dots, B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n+2^{n-2}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n+2^{n-2}p_n} := M^n_2,\ B^{(n,n)}_{j_{(n,n-1)}+2^{n-1}p_n+2^{n-2}p_n+1} := I, \dots, B^{(n,n)}_{j_{(n,n-1)}+2^np_n-1} := I,$$

$$\vdots$$

$$B^{(n,n)}_{j_{(n,n-1)}+(l_n-1)2^{n-1}p_n} := I,\ B^{(n,n)}_{j_{(n,n-1)}+(l_n-1)2^{n-1}p_n+1} := I,$$
$$\dots, B^{(n,n)}_{j_{(n,n-1)}+(l_n-1)2^{n-1}p_n+2^{n-2}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+(l_n-1)2^{n-1}p_n+2^{n-2}p_n} := M^n_{l_n},\ B^{(n,n)}_{j_{(n,n-1)}+(l_n-1)2^{n-1}p_n+2^{n-2}p_n+1} := I,$$
$$\dots, B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n} := I,\ B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n+1} := I,$$
$$\dots, B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n+2^{n-2}p_n-1} := I,$$

$$B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n+2^{n-2}p_n} := I,\ B^{(n,n)}_{j_{(n,n-1)}+l_n2^{n-1}p_n+2^{n-2}p_n+1} := I,$$
$$\dots, B^{(n,n)}_{j_{(n,n-1)}+(l_n+1)2^{n-1}p_n-1} := I,$$

$$\vdots$$

$$B^{(n,n)}_{j_{(n,n-1)}+r_n-r_{n-1}-1} := I, \dots, B^{(n,n)}_{j_{(n,n-1)}(r_n-r_{n-1})-1} := I,$$

where (see (5.22))

$$r_n - r_{n-1} = r_{n-1}\left[2^{n-1}l_n - 1\right] \geq p_{n-1}2^n\left[2^{n-1}l_n - 1\right] = 2p_n\left[2^{n-1}l_n - 1\right] > l_n2^{n-1}p_n.$$

Finally, in the n-th step, we define

$$B^n_k := B^{(1,n)}_k \cdot B^{(2,n)}_k \cdots B^{(n,n)}_k, \quad k \in \mathbb{Z}, \tag{5.48}$$

and

$$\tilde{B}^n_k := A_k \cdot B^1_k \cdot B^2_k \cdots B^n_k, \quad k \in \mathbb{Z}.$$

Then, we consider the solution $\{x^n_k\}_{k\in\mathbb{Z}}$ of

$$x_{k+1} = \tilde{B}^n_k \cdot x_k, \quad k \in \mathbb{Z}, \quad x_0 = u.$$

Applying Lemma 5.1 for $B^{n+j}_k = I$, $k \in \mathbb{Z}$, $j \in \mathbb{N}$, it suffices to consider the case, when $\{x^n_k\}$ is almost periodic, and to continue in the construction.

All sequence $\{B^n_k\}_{k\in\mathbb{Z}}$ is periodic as the product of n periodic sequences. Let q_n be a period of $\{B^n_k\}$, $n \in \mathbb{N}$. In the construction, we can obtain matrices different from I only for

$$B^1_{2l+1}, B^{(1,2)}_{4l+2}, B^{(2,2)}_{8l+4}, \dots, B^{(1,n)}_{lp_n+\frac{p_n}{2}}, B^{(2,n)}_{2lp_n+p_n}, \dots, B^{(n,n)}_{2^{n-1}lp_n+2^{n-2}p_n}, \dots, \tag{5.49}$$

where $l \in \mathbb{Z}$. Considering (5.38) and (5.48) (see also (5.35)), the structure of the indices of matrices in (5.49) gives (5.3). It is seen that (5.1) and (5.4) follow from (5.19) and from the construction. Analogously, (5.2) follows from (5.20). Thus, applying Lemma 5.1 for the sequence of

$$B_k := A_k \cdot B^1_k \cdot B^2_k \cdots B^n_k \cdots, \quad k \in \mathbb{Z},$$

we have that $\{B_k\} \in \mathcal{O}^\sigma_\varepsilon(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$.

To complete the proof, it suffices to show that the solution $\{x_k\}_{k\in\mathbb{Z}}$ of the problem

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

is not almost periodic. On contrary, let us assume that $\{x_k\}$ is almost periodic. We use Theorem 3.6 for $l_1 = 0$, $l_{n+1} = r_n$, $n \in \mathbb{N}$ (see (5.39)). We know that, for any $\xi > 0$, there exist infinitely many $i, j \in \mathbb{N}$ satisfying

$$\left\| x_{k+l_i} - x_{k+l_j} \right\| < \xi, \quad k \in \mathbb{Z}. \tag{5.50}$$

From the construction (consider (5.30), (5.33), ..., (5.40), (5.43), ..., (5.46)), we obtain

$$B_k^{n+j} = I, \quad k \in \{0, 1, \dots, q_n - 1\},\ n, j \in \mathbb{N}. \tag{5.51}$$

Hence, we get

$$\begin{aligned}
\left\| x_{j(1,0)} - x_{j(1,0)+r_1} \right\| &= \left\| B_{j(1,0)-1} \cdots B_1 \cdot B_0 \cdot u - B_{j(1,0)+r_1-1} \cdots B_1 \cdot B_0 \cdot u \right\| \\
&= \| B^1_{j(1,0)-1} \cdots B^1_1 \cdot B^1_0 \cdot A_{j(1,0)-1} \cdots A_1 \cdot A_0 \cdot u \\
&\quad - B^1_{j(1,0)+r_1-1} \cdots B^1_1 \cdot B^1_0 \cdot A_{j(1,0)+r_1-1} \cdots A_1 \cdot A_0 \cdot u \| \\
&= \left\| B^1_{j(1,0)-1} \cdots B^1_1 \cdot B^1_0 \cdot x^0_{j(1,0)} - B^1_{j(1,0)+r_1-1} \cdots B^1_1 \cdot B^1_0 \cdot x^0_{j(1,0)+r_1} \right\|,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left\| x_{j(1,0)} - x_{j(1,0)+r_1} \right\| \\
&\quad = \left\| B^1_{j(1,0)-1} \cdots B^1_1 \cdot B^1_0 \cdot x^0_{j(1,0)} - B^1_{j(1,0)+r_1-1} \cdots B^1_1 \cdot B^1_0 \cdot x^0_{j(1,0)+r_1} \right\|.
\end{aligned} \tag{5.52}$$

If (5.27) is valid, then we can rewrite (5.52) into

$$\left\| x_{j(1,0)} - x_{j(1,0)+r_1} \right\| = \left\| I \cdots I \cdot I \cdot x^0_{j(1,0)} - I \cdots I \cdot I \cdot x^0_{j(1,0)+r_1} \right\| \geq \frac{\vartheta_1}{2}.$$

If (5.28) is true, then we have

$$\left\| x_{j(1,0)} - x_{j(1,0)+r_1} \right\| = \left\| I \cdots I \cdot I \cdot x^0_{j(1,0)} - M^1_{l_1} \cdots M^1_2 \cdot M^1_1 \cdot x^0_{j(1,0)+r_1} \right\| > \frac{\zeta}{2} \geq \frac{\vartheta_1}{2},$$

which follows from (5.21), (5.24), (5.25), and from the inequality (see (5.26), (5.28))

$$\left\| u - x^0_{j(1,0)+r_1} \right\| \leq \left\| u - x^0_{j(1,0)} \right\| + \left\| x^0_{j(1,0)} - x^0_{j(1,0)+r_1} \right\| < \frac{\vartheta_1}{2} + \frac{\vartheta_1}{2} = \vartheta_1.$$

In the both cases, we get

$$\left\| x_{j(1,0)} - x_{j(1,0)+r_1} \right\| \geq \frac{\vartheta_1}{2}. \tag{5.53}$$

Considering (5.51) and the construction, we can express

$$\begin{aligned}
\left\| x_{j(1,1)} - x_{j(1,1)+r_2} \right\| &= \left\| B_{j(1,1)-1} \cdots B_1 \cdot B_0 \cdot u - B_{j(1,1)+r_2-1} \cdots B_1 \cdot B_0 \cdot u \right\| \\
&= \left\| B^{(1,2)}_{j(1,1)-1} \cdots B^{(1,2)}_1 \cdot B^{(1,2)}_0 \cdot \tilde{B}^1_{j(1,1)-1} \cdots \tilde{B}^1_1 \cdot \tilde{B}^1_0 \cdot u \right. \\
&\qquad \left. - B^{(1,2)}_{j(1,1)+r_2-1} \cdots B^{(1,2)}_1 \cdot B^{(1,2)}_0 \cdot \tilde{B}^1_{j(1,1)+r_2-1} \cdots \tilde{B}^1_1 \cdot \tilde{B}^1_0 \cdot u \right\| \\
&= \left\| B^{(1,2)}_{j(1,1)-1} \cdots B^{(1,2)}_1 \cdot B^{(1,2)}_0 \cdot x^1_{j(1,1)} \right. \\
&\qquad \left. - B^{(1,2)}_{j(1,1)+r_2-1} \cdots B^{(1,2)}_1 \cdot B^{(1,2)}_0 \cdot x^1_{j(1,1)+r_2} \right\|,
\end{aligned}$$

i.e.,

$$\begin{aligned}
&\left\| x_{j(1,1)} - x_{j(1,1)+r_2} \right\| \\
&\quad = \left\| B^{(1,2)}_{j(1,1)-1} \cdots B^{(1,2)}_{1} \cdot B^{(1,2)}_{0} \cdot x^{1}_{j(1,1)} - B^{(1,2)}_{j(1,1)+r_2-1} \cdots B^{(1,2)}_{1} \cdot B^{(1,2)}_{0} \cdot x^{1}_{j(1,1)+r_2} \right\|.
\end{aligned} \tag{5.54}$$

If (5.31) is valid, then (5.54) takes the form

$$\left\| x_{j(1,1)} - x_{j(1,1)+r_2} \right\| = \left\| I \cdots I \cdot I \cdot x^{1}_{j(1,1)} - I \cdots I \cdot I \cdot x^{1}_{j(1,1)+r_2} \right\| \geq \frac{\vartheta_2}{2}. \tag{5.55}$$

If (5.31) is not valid, then we have

$$\left\| x_{j(1,1)} - x_{j(1,1)+r_2} \right\| = \left\| I \cdots I \cdot I \cdot x^{1}_{j(1,1)} - M^{2}_{l_2} \cdots M^{2}_{2} \cdot M^{2}_{1} \cdot x^{1}_{j(1,1)+r_2} \right\| > \frac{\zeta}{2} \geq \frac{\vartheta_2}{2}. \tag{5.56}$$

Indeed, it suffices to consider (5.21), (5.24), (5.25), and the inequality (see also (5.29))

$$\left\| u - x^{1}_{j(1,1)+r_2} \right\| \leq \left\| u - x^{1}_{j(1,1)} \right\| + \left\| x^{1}_{j(1,1)} - x^{1}_{j(1,1)+r_2} \right\| < \frac{\vartheta_2}{2} + \frac{\vartheta_2}{2} = \vartheta_2.$$

Again, one can express

$$\begin{aligned}
\left\| x_{j(2,1)} - x_{j(2,1)+r_2-r_1} \right\| &= \left\| B_{j(2,1)-1} \cdots B_1 \cdot B_0 \cdot u - B_{j(2,1)+r_2-r_1-1} \cdots B_1 \cdot B_0 \cdot u \right\| \\
&= \Big\| B^{(2,2)}_{j(2,1)-1} \cdots B^{(2,2)}_{1} \cdot B^{(2,2)}_{0} \cdot \tilde{B}^{(1,2)}_{j(2,1)-1} \cdots \tilde{B}^{(1,2)}_{1} \cdot \tilde{B}^{(1,2)}_{0} \cdot u \\
&\qquad - B^{(2,2)}_{j(2,1)+r_2-r_1-1} \cdots B^{(2,2)}_{0} \cdot \tilde{B}^{(1,2)}_{j(2,1)+r_2-r_1-1} \cdots \tilde{B}^{(1,2)}_{0} \cdot u \Big\| \\
&= \Big\| B^{(2,2)}_{j(2,1)-1} \cdots B^{(2,2)}_{1} \cdot B^{(2,2)}_{0} \cdot x^{(1,2)}_{j(2,1)} \\
&\qquad - B^{(2,2)}_{j(2,1)+r_2-r_1-1} \cdots B^{(2,2)}_{1} \cdot B^{(2,2)}_{0} \cdot x^{(1,2)}_{j(2,1)+r_2-r_1} \Big\|,
\end{aligned}$$

i.e.,

$$\begin{aligned}
\left\| x_{j(2,1)} - x_{j(2,1)+r_2-r_1} \right\| &= \Big\| B^{(2,2)}_{j(2,1)-1} \cdots B^{(2,2)}_{1} \cdot B^{(2,2)}_{0} \cdot x^{(1,2)}_{j(2,1)} \\
&\qquad - B^{(2,2)}_{j(2,1)+r_2-r_1-1} \cdots B^{(2,2)}_{1} \cdot B^{(2,2)}_{0} \cdot x^{(1,2)}_{j(2,1)+r_2-r_1} \Big\|.
\end{aligned} \tag{5.57}$$

If (5.34) is valid, then (5.57) gives

$$\left\| x_{j(2,1)} - x_{j(2,1)+r_2-r_1} \right\| = \left\| I \cdots I \cdot I \cdot x^{(1,2)}_{j(2,1)} - I \cdots I \cdot I \cdot x^{(1,2)}_{j(2,1)+r_2-r_1} \right\| \geq \frac{\vartheta_2}{2}. \tag{5.58}$$

If (5.34) is not valid, then (5.57) gives

$$\begin{aligned}
&\left\| x_{j(2,1)} - x_{j(2,1)+r_2-r_1} \right\| \\
&\quad = \left\| I \cdots I \cdot I \cdot x^{(1,2)}_{j(2,1)} - M^{2}_{l_2} \cdots M^{2}_{2} \cdot M^{2}_{1} \cdot x^{(1,2)}_{j(2,1)+r_2-r_1} \right\| > \frac{\zeta}{2} \geq \frac{\vartheta_2}{2},
\end{aligned} \tag{5.59}$$

where (5.21), (5.24), (5.25), (5.32), and (5.34) are used.

Finally (see (5.55), (5.56), (5.58), and (5.59)), from the second step of the construction, we have
$$\left\| x_{j(1,1)} - x_{j(1,1)+r_2} \right\| \geq \frac{\vartheta_2}{2}, \quad \left\| x_{j(2,1)} - x_{j(2,1)+r_2-r_1} \right\| \geq \frac{\vartheta_2}{2}. \tag{5.60}$$
Analogously as (5.53) and (5.60) (consider again (5.21), (5.24), (5.25) and the construction with (5.37), (5.41), (5.42), (5.44), ..., (5.45), (5.47)), one can obtain
$$\left\| x_{j_{(1,n-1)}} - x_{j_{(1,n-1)}+r_n} \right\| \geq \frac{\vartheta_n}{2},$$
$$\left\| x_{j_{(2,n-1)}} - x_{j_{(2,n-1)}+r_n-r_1} \right\| \geq \frac{\vartheta_n}{2},$$
$$\vdots$$
$$\left\| x_{j_{(n,n-1)}} - x_{j_{(n,n-1)}+r_n-r_{n-1}} \right\| \geq \frac{\vartheta_n}{2}$$
for all $n \in \mathbb{N}$.

Considering Lemma 5.2, we can assume that (see (5.23) and (5.24))
$$\sup_{j\in\mathbb{N}} K_j < \infty, \quad \text{i.e.,} \quad \vartheta := \inf_{j\in\mathbb{N}} \vartheta_j > 0. \tag{5.61}$$
Thus, for all $n \in \mathbb{N}$, we obtain
$$\left\| x_{j_{(1,n-1)}} - x_{j_{(1,n-1)}+r_n} \right\| \geq \frac{\vartheta}{2},$$
$$\left\| x_{j_{(2,n-1)}} - x_{j_{(2,n-1)}+r_n-r_1} \right\| \geq \frac{\vartheta}{2},$$
$$\vdots$$
$$\left\| x_{j_{(n,n-1)}} - x_{j_{(n,n-1)}+r_n-r_{n-1}} \right\| \geq \frac{\vartheta}{2}.$$
Especially, for all $i \neq j$, $i, j \in \mathbb{N}$, there exists $l \in \mathbb{Z}$ such that
$$\left\| x_{l+l_i} - x_{l+l_j} \right\| \geq \frac{\vartheta}{2}.$$
This contradiction (consider (5.50) for $2\xi \leq \vartheta$) proves that $\{x_k\}$ is not almost periodic. □

Remark 5.4. It is seen that the statement of Theorem 5.3 does not change if we replace system $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ by a periodic one. Indeed, it follows directly from Definition 3.1.

Remark 5.5. To illustrate Theorem 5.3, let us consider an arbitrary periodic system $\{M_k\}$ in the complex case (i.e., for $F = \mathbb{C}$ with the usual absolute value). It means that we have a system
$$x_{k+1} = M_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad \text{where} \quad M_k = M_{k+p}, \quad k \in \mathbb{Z},$$
for a positive integer p and arbitrarily given non-singular complex matrices $M_0, M_1, \ldots, M_{p-1}$. It is well-known that a solution of $\{M_k\}$ is almost periodic if and only if it is bounded (see, e.g., [33, Corollary 3.9] or [35, Theorem 5]). The fundamental matrix $\Phi(k, 0)$ of $\{M_k\}$ satisfying $\Phi(0, 0) = I$ is given by
$$\Phi(lp + i, 0) = M_{i-1} \cdots M_1 \cdot M_0 \cdot \left(M_{p-1} \cdots M_1 \cdot M_0\right)^l, \quad l \in \mathbb{N} \cup \{0\}, i \in \{1, \ldots, p\}.$$

Thus, to describe the structure of almost periodic solutions, it suffices to consider the multiples $(M_{p-1} \cdots M_1 \cdot M_0)^l$ and, in fact, the constant system

$$x_{k+1} = M_{p-1} \cdots M_1 \cdot M_0 \cdot x_k, \quad k \in \mathbb{Z}.$$

For any constant system given by a non-singular complex matrix M, one can easily find a commutative matrix group $\mathcal{X}$ containing M and having property P with respect to a vector (e.g., one can consider the group generated by matrices cM for all complex numbers $c = \sin l + \mathrm{i} \cos l$, $l \in \mathbb{Z}$). Applying Theorem 5.3, we know that, in any neighbourhood of the considered system, there exists a limit periodic system whose coefficient matrices are from the group and whose fundamental matrix is not almost periodic. In addition, such a limit periodic system can be found for any commutative group $\mathcal{X}$ which contains M and which has property P with respect to at least one vector.

Remark 5.6. We repeat that the basic motivation of this paper comes from [35], where non-asymptotically almost periodic solutions of limit periodic systems are considered. Of course, systems with coefficient matrices from bounded groups are analysed in [35]. For general groups, it is not possible to prove the main results of [35], i.e., Theorems 4.2 and 4.3. It suffices to consider the constant system given by matrix $I/2$ in the complex case. Any solution $\{x_k\}_{k\in\mathbb{Z}}$ of this system has the property that

$$\| x_{l+1} \| = \frac{\| x_l \|}{2}, \quad l \in \mathbb{N}.$$

Thus, there exists a neighbourhood of the system such that, for any solution $\{y_k\}_{k\in\mathbb{Z}}$ of an almost periodic system from the neighbourhood, we obtain $\lim_{k\to\infty} \| y_k \| = 0$, which gives the asymptotic almost periodicity of $\{y_k\}$ (see Remark 3.10).

At the same time, in [35], there is required that the studied matrix group has property P. Since the group $\mathcal{X}$ has property P only with respect to one vector in the statement of Theorem 5.3, we can apply this theorem for groups of matrices in the following form

$$\begin{pmatrix} X & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{pmatrix},$$

where X is taken from a commutative matrix group having property P with respect to a concrete vector. In this sense, Theorem 5.3 generalizes Theorem 4.2 as well.

The construction from the proof of Theorem 5.3 can be applied for the Cauchy (initial) problem. Especially, we immediately obtain the following result.

Theorem 5.7. *Let a non-zero vector $u \in F^m$ be given. Let $\mathcal{X}$ have the property that there exist $\zeta > 0$ and $K > 0$ such that, for all $\delta > 0$, one can find matrices $M_1, M_2, \ldots, M_l \in \mathcal{X}$ satisfying*

$$M_i \in \mathcal{O}^{\varrho}_{\delta}(I),\ i \in \{1, 2, \ldots, l\}, \quad \| M_l \cdots M_2 \cdot M_1 \cdot u - u \| > \zeta, \quad \| M_l \cdots M_2 \cdot M_1 \| < K.$$

For any $\{A_k\} \in \mathcal{LP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}^{\sigma}_{\varepsilon}(\{A_k\}) \cap \mathcal{LP}(\mathcal{X})$ for which the solution of

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

is not almost periodic.

Proof. The theorem follows from the proof of Theorem 5.3, where (5.61) is satisfied (i.e., the case, which is covered by Lemma 5.2, does not happen). □

Similarly to Theorem 4.3 which is the almost periodic version of Theorem 4.2, we formulate the below given Theorem 5.10 as the almost periodic version of Theorem 5.3. To prove it, we need the next two lemmas.

Lemma 5.8. *Let $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$ be arbitrarily given. Let $\{\delta_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ be a decreasing sequence satisfying (5.1) and let $\{B_k^n\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ be periodic sequences for $n \in \mathbb{N}$ such that (5.2) and (5.3) are valid. Then, $\{B_k\} \in \mathcal{AP}(\mathcal{X})$ if*

$$B_k := A_k \cdot B_k^1 \cdot B_k^2 \cdots B_k^n \cdots, \quad k \in \mathbb{Z}.$$

In addition, $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$ if (5.4) is fulfilled.

Proof. The lemma can be proved analogously as Lemma 5.1. In the proof of Lemma 5.1, it suffices to put $C_k^n = A_k$ for all $k \in \mathbb{Z}$, $n \in \mathbb{N}$, to use Theorem 3.5, and to consider the almost periodicity of $\{C_k^n \cdot B_k^1 \cdot B_k^2 \cdots B_k^n\}_{k\in\mathbb{Z}} \subset \mathcal{X}$ which follows from Theorem 3.6, Lemma 4.4, and from Remark 3.4. □

Using the same way which is applied in the proof of Lemma 5.2, we can prove its almost periodic counterpart. Indeed, we do not use the limit periodicity of $\{A_k\}$ in the proof (consider also Lemma 5.8).

Lemma 5.9. *If for any $\delta > 0$ and $K > 0$, there exist matrices $M_1, M_2, \ldots, M_l \in \mathcal{X}$ such that*

$$M_i \in \mathcal{O}_\delta^\varrho(I), \ i \in \{1, 2, \ldots, l\}, \quad \| M_l \cdots M_2 \cdot M_1 \| > K,$$

then, for any $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$ whose fundamental matrix is not almost periodic.

Theorem 5.10. *Let $\mathcal{X}$ have property P with respect to a vector. For any $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$ whose fundamental matrix is not almost periodic.*

Proof. The theorem can be proved using the same construction as Theorem 5.3. It suffices to replace Lemma 5.1 by Lemma 5.8 and Lemma 5.2 by Lemma 5.9. □

Analogously, we get the following result as well.

Theorem 5.11. *Let a non-zero vector $u \in F^m$ be given. Let $\mathcal{X}$ have the property that there exist $\zeta > 0$ and $K > 0$ such that, for all $\delta > 0$, one can find matrices $M_1, M_2, \ldots, M_l \in \mathcal{X}$ satisfying*

$$M_i \in \mathcal{O}_\delta^\varrho(I), \ i \in \{1, 2, \ldots, l\}, \quad \| M_l \cdots M_2 \cdot M_1 \cdot u - u \| > \zeta, \quad \| M_l \cdots M_2 \cdot M_1 \| < K.$$

For any $\{A_k\} \in \mathcal{AP}(\mathcal{X})$ and $\varepsilon > 0$, there exists a system $\{B_k\} \in \mathcal{O}_\varepsilon^\sigma(\{A_k\})$ for which the solution of

$$x_{k+1} = B_k \cdot x_k, \quad k \in \mathbb{Z}, \qquad x_0 = u$$

is not almost periodic.

Remark 5.12. We add that Theorems 5.10 and 5.11 do not follow from Theorems 5.3 and 5.7. Indeed, in [5], there is proved that there exist systems which are almost periodic and which are not limit periodic (e.g., the sequence $\{e^{ik}\}_{k\in\mathbb{Z}}$ is almost periodic and, at the same time, it is not limit periodic). It means that there exist almost periodic systems which have neighbourhoods without limit periodic systems.

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